Dynamical Systems (ME221)

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

Week 13: Recap

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Final Exam

- Exam Date: 4th of July, Tuesday

Exam Format and Rules

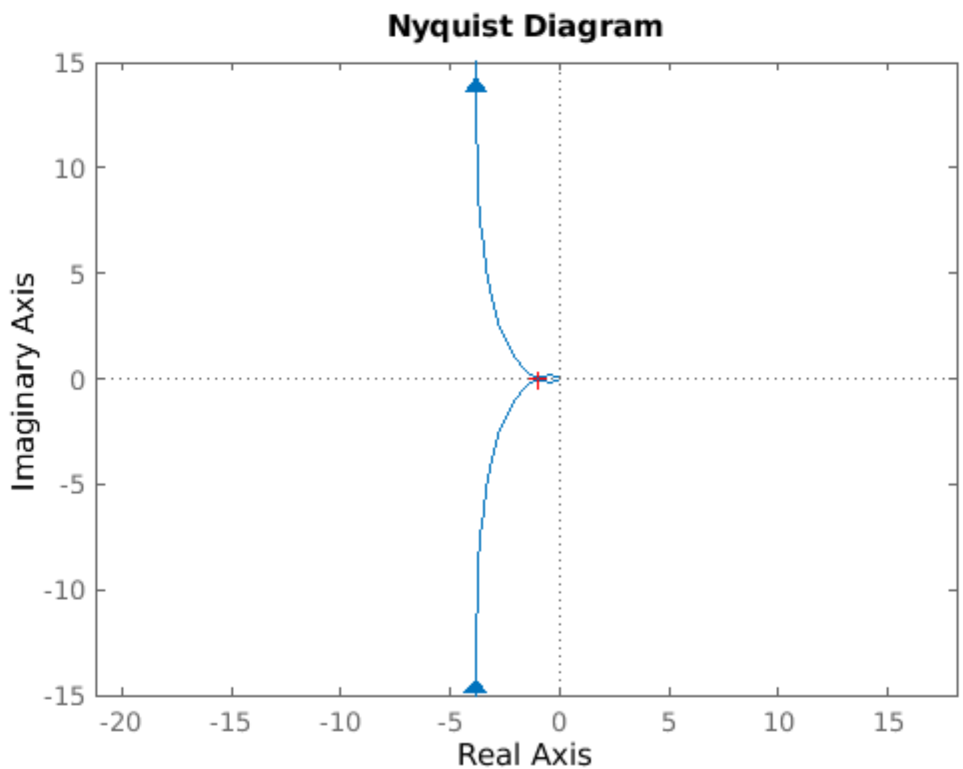
- Hand-written Notes: A4 page, double-sided
- Basic Calculator (non-programmable)
- We will provide Laplace Transform Tables
- Check Moodle for details and announcements

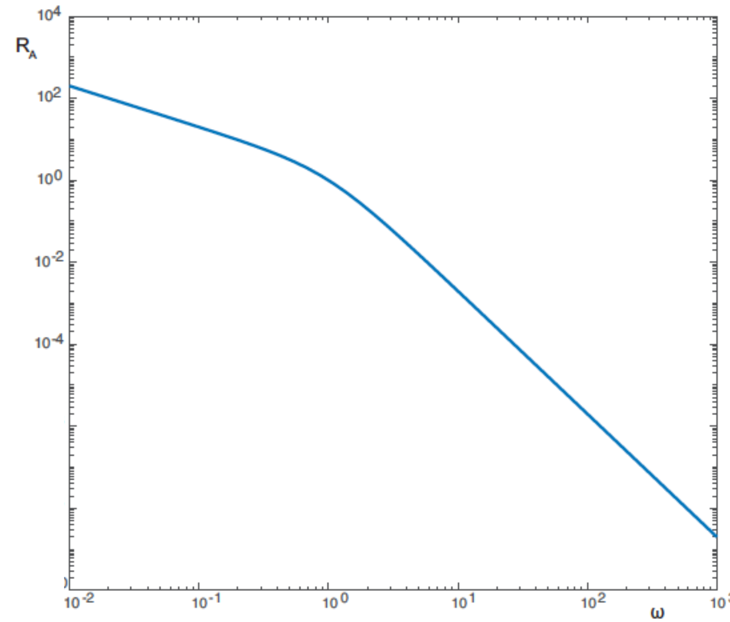
Extra Materials

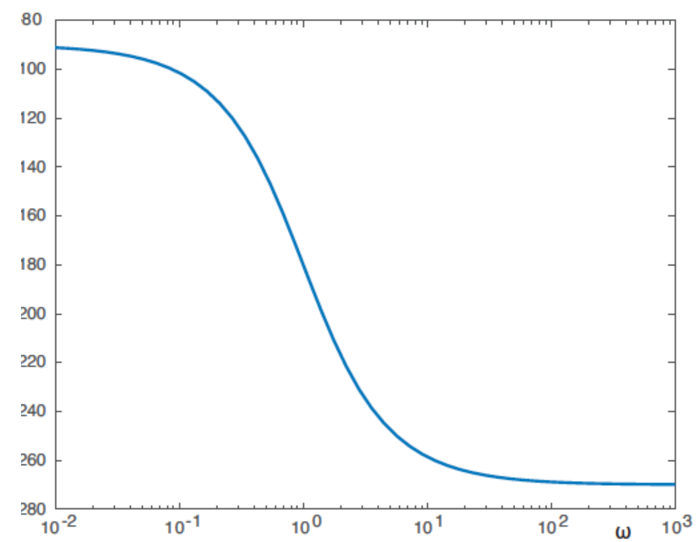
- Previous exam problems
- Solved problems in Polycopie

Nyquist Plots (PS 11)

$$G(s) = \frac{2}{s(s+1)^2}$$

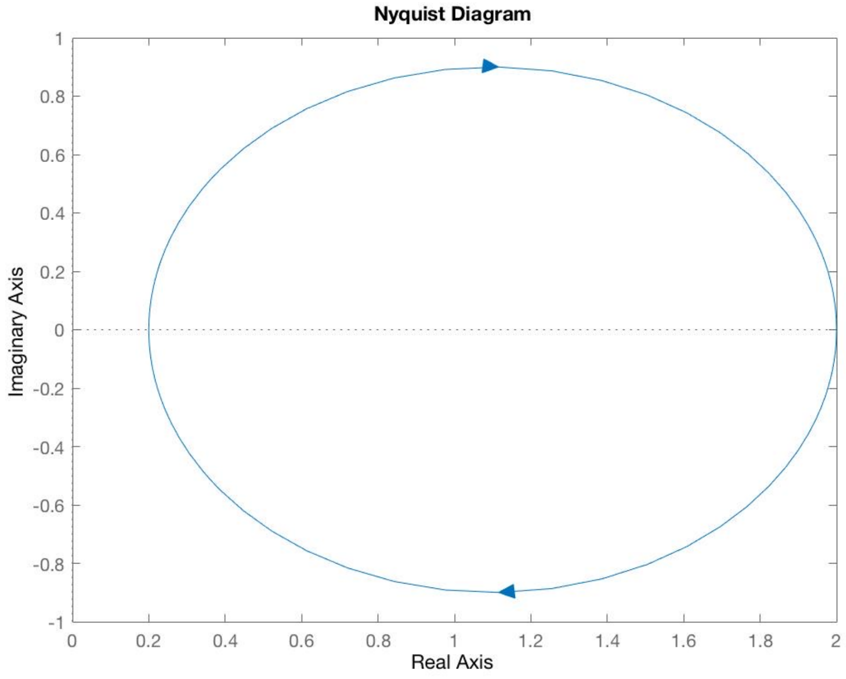


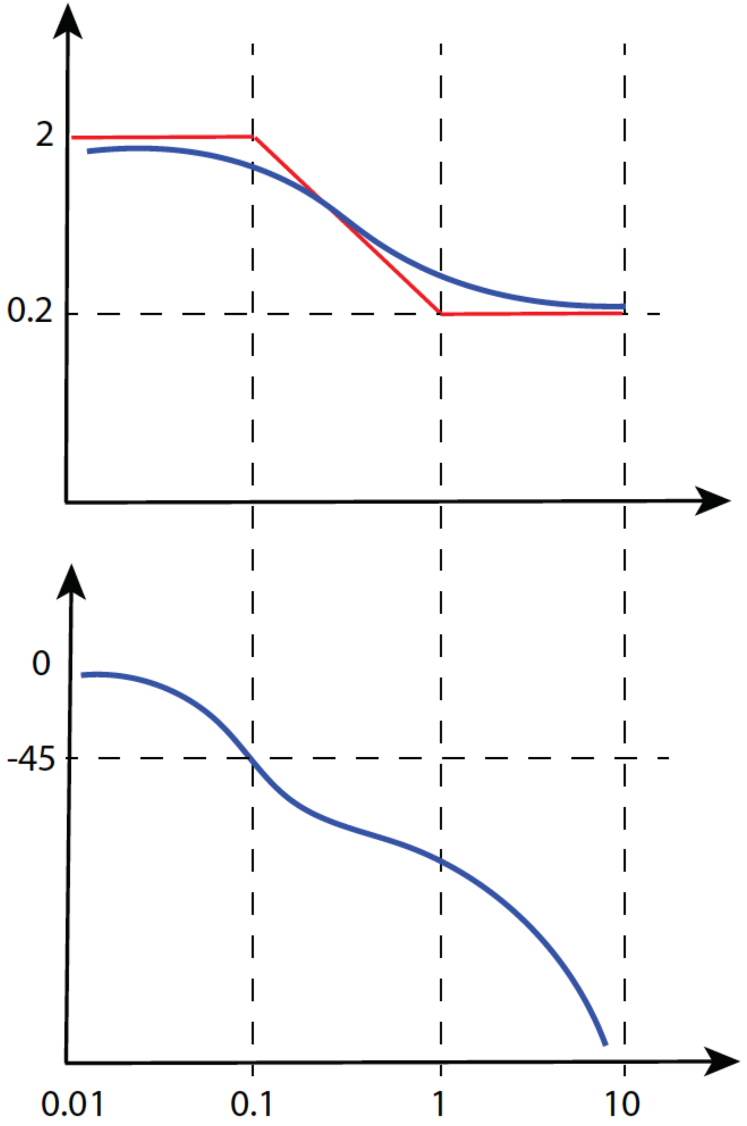




Nyquist Plots (PS 11)

$$G(s) = \frac{2(s+1)}{10s+1}e^{-2s}$$





Model-driven Questions

- **Mathematical Modeling** (P2-P3)
- **Linearization** (P4)
  - Properties of Linear and Time-Invariant Systems (P1)

  - **State space representation** (P4)
    - Time-domain Analysis

  - **Transfer Function: Laplace Transform** (P6)
    - Inverse Laplace Transform for Time-domain Analysis (P7)
    - **Sinusoidal Transfer Function** for Frequency Domain Analysis (P9)
    - **Poles and Zeros**: Stability (P8)

  - Impulse Response (P5)
    - **Convolution Operation:** Time-domain Analysis

  - **Generating Diagrams:** Output vs time, Bode Plot, Nyquist Plot (P10)

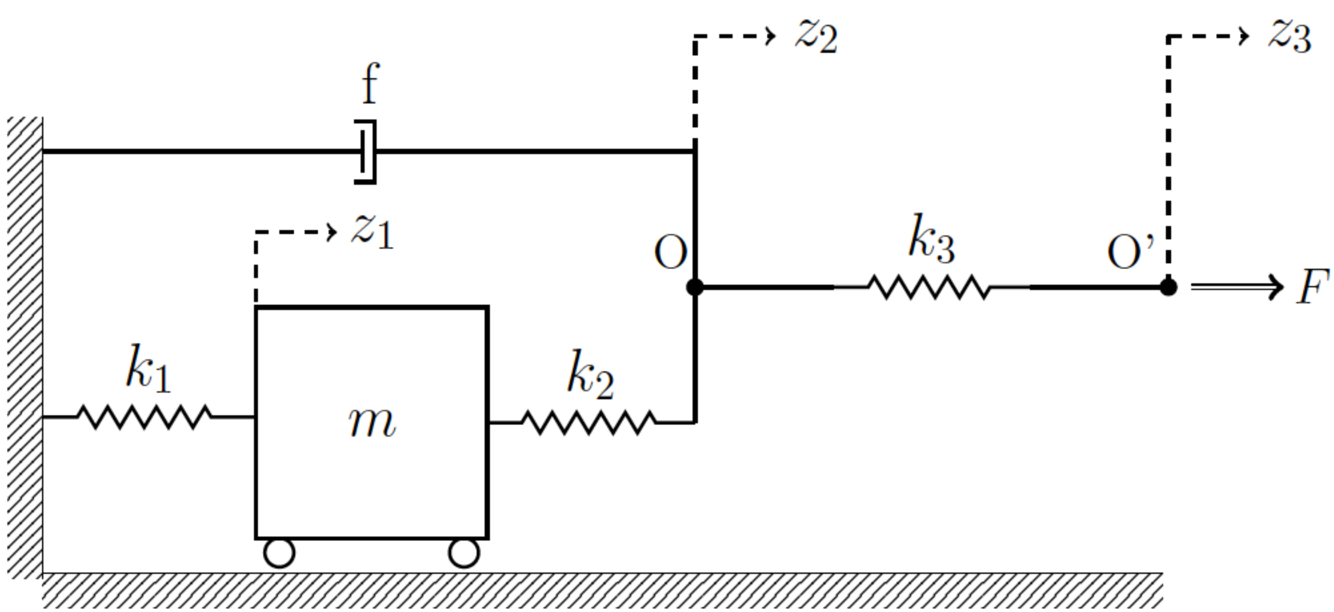
Data-driven Questions

- Extract information from data and plots (P10)
    - **Unit-step response:** Time-domain analysis
        - **Settling time**, **rise time**, **peak time**, **overshoot, damping**
        - Transient response
        - Find transfer function
            - Switch to frequency response
        - Reconstruct mathematical model and identify the system

    - Bode plots
        - **Corner frequency**, **resonant frequency**
        - Steady-state response
        - **Design of filters**
        - Find transfer function
            - Switch to time domain response
        - Reconstruct mathematical model and identify the system

Problem 1 (Modeling)

- System is initially at rest.



- Equations of motion
- State-space representation
- Transfer function
- Analogous circuit

Problem 1: Notes

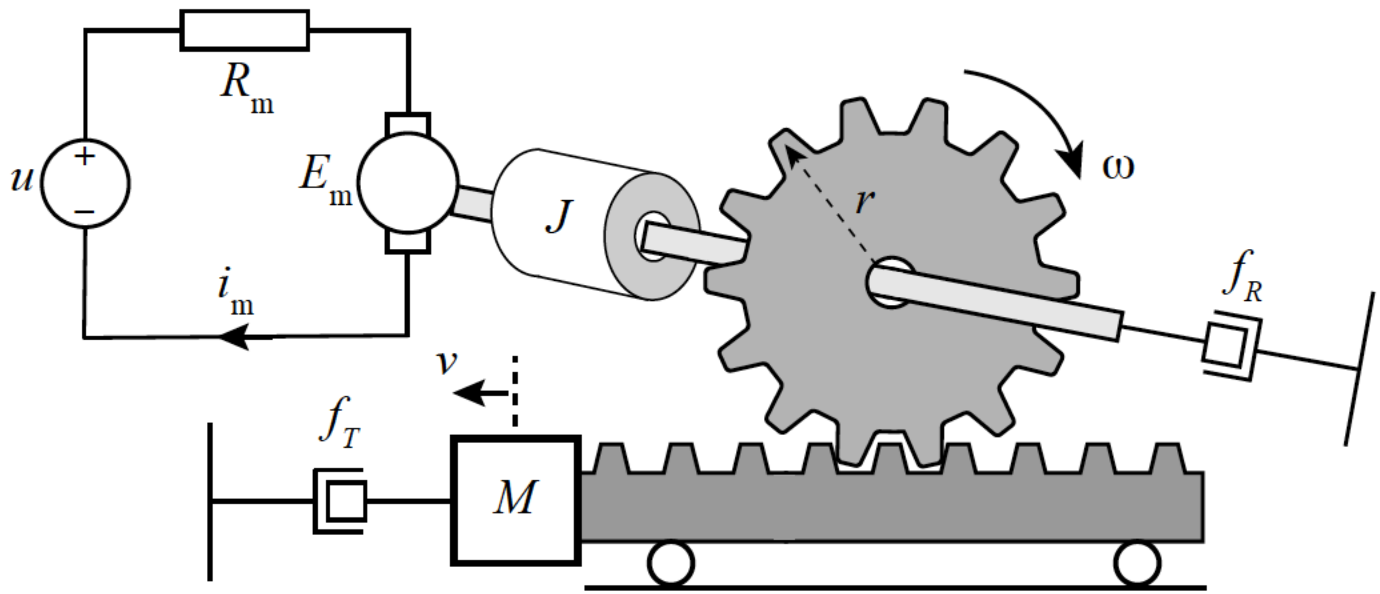
- System: Mechanical, electrical, or electromechanical
- Pay attention to definitions of parameters and variables
- **What assumptions did we make to simplify the problem?**
- Analogous circuit: Switching between systems

Problem 1 (Modeling)

- System is initially at rest.

$$E_m(t) = K_m \omega(t)$$

$$T_m = K_t i(t)$$



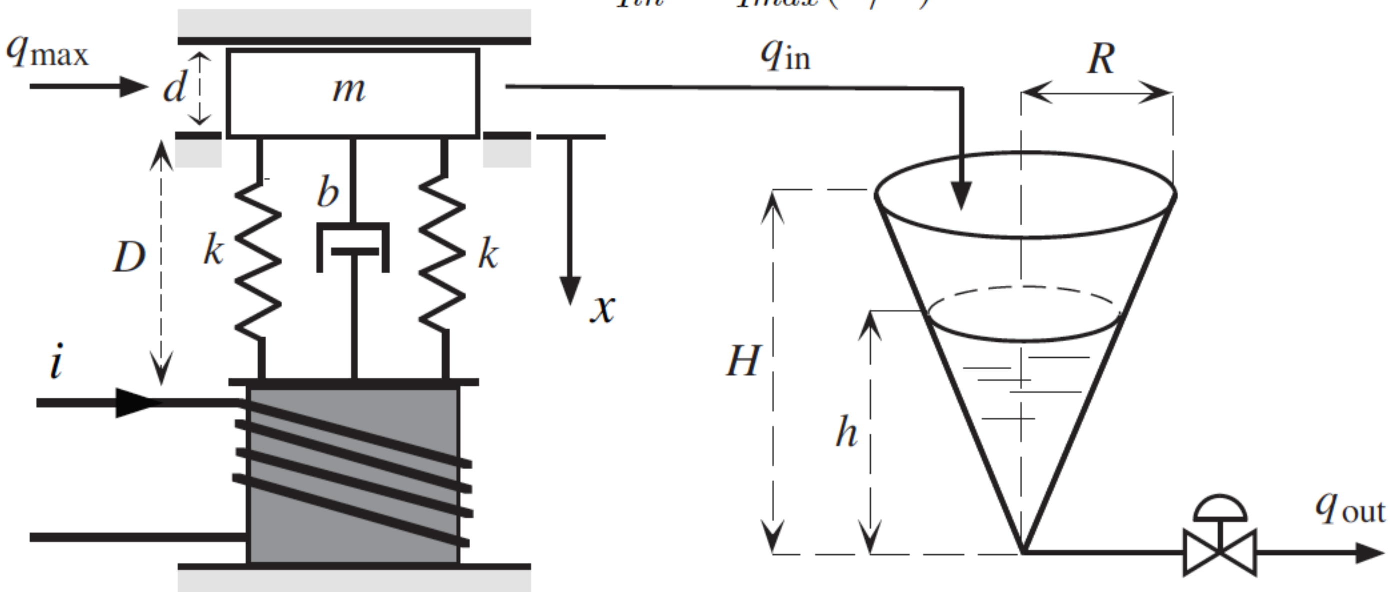
- Equations of motion
- Transfer function
- Unit-step response
- Calculate the output for a given input (Laplace)

Problem 1 (Modeling)

- System is initially at rest.

$$F(x,i) = \frac{L}{2}\frac{i^2}{(D-x)^2}$$

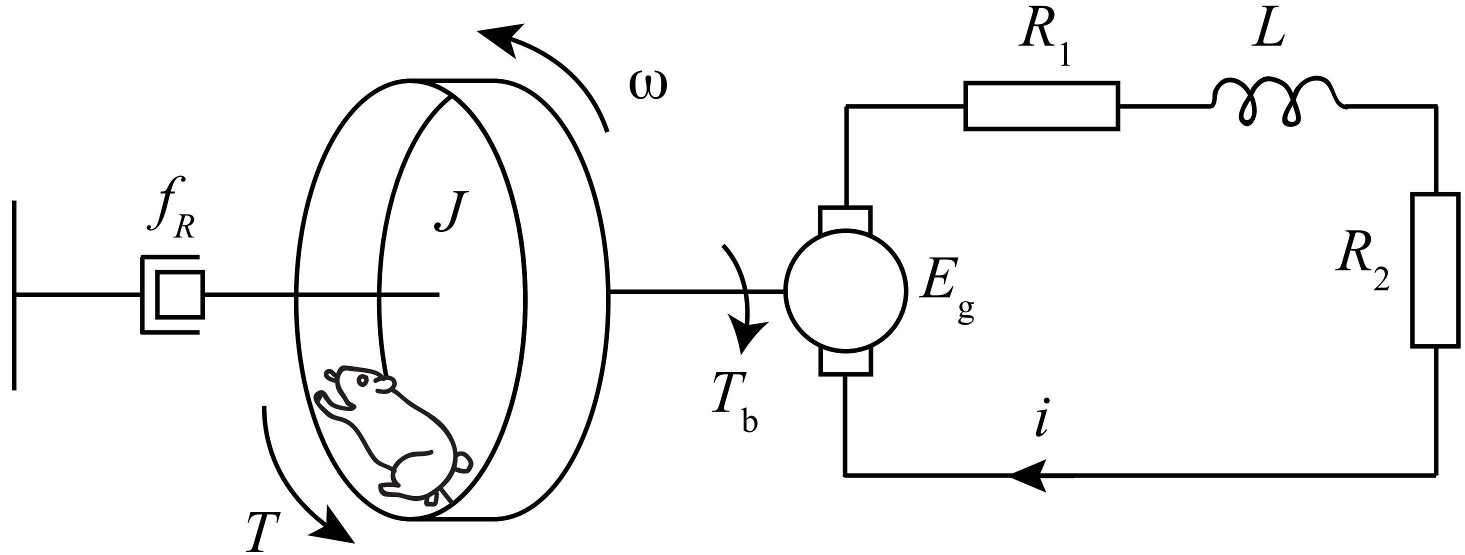
$$q_{in} = q_{max}(x/d)$$



$$q_{out} = c\sqrt{h(t)}$$

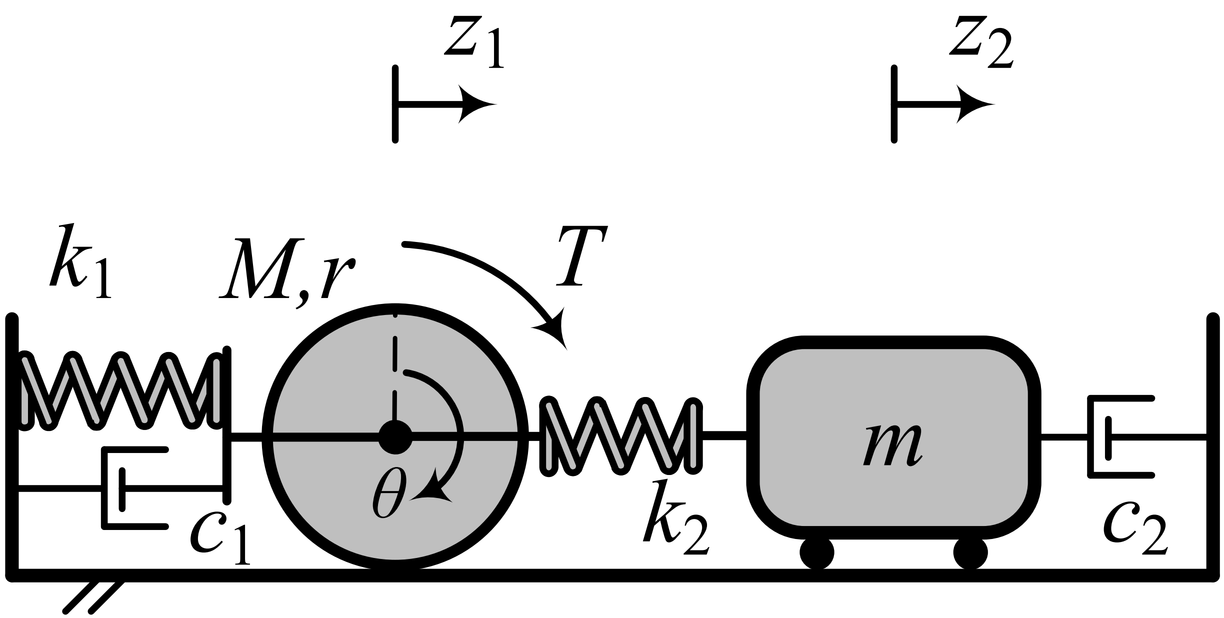
- Equations of motion (PS3 Ex4 and PS4 Ex3)
- Linearize the system around a given equilibrium point
- State-space representation

Problem 1 (Modeling)



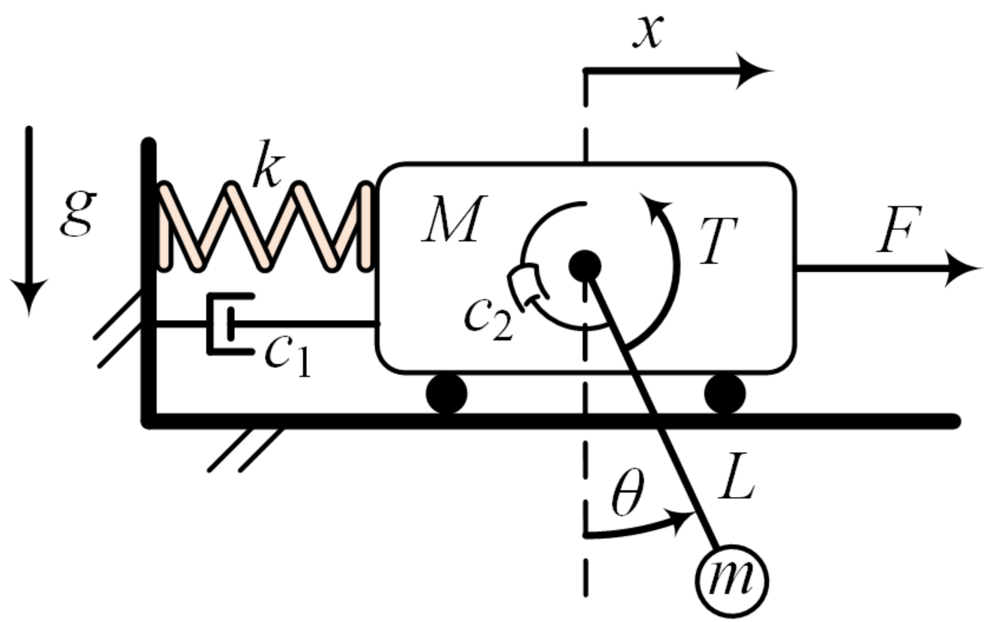
- Equations of motion
- State-space representation
- Transfer function

Problem 1 (Modeling)



- Equations of motion
- State-space representation
- Transfer function

Problem 1 (Modeling)



- Equations of motion
- State-space representation
- Transfer function

Problem 1 (Linearization)

- Nonlinear System

$$\dot{x}_1 = 2x_1 + 2x_1x_2 + u \qquad x_1(0) = 0$$
$$\dot{x}_2 = x_2 + 3x_1x_2 \qquad x_2(0) = 0$$

- Linearize around a given equilibrium point

$$\bar{u} = 1 \qquad \bar{x}_1,\, \bar{x}_2 \neq 0$$

- **Get comfortable with the Jacobian linearization approach**

Equilibrium Point (Lecture 4 Slide 25)

$$\dot{x}(t) = f[x(t), u(t)]$$

$$y(t) = g[x(t), u(t)]$$

$$x(0) = x_0$$

- At the equilibrium point $(\bar{u}, \bar{x}, \bar{y})$, the derivatives will go to zero.

$$0 = f[\bar{x}, \bar{u}]$$

$$\bar{y} = g[\bar{x}, \bar{u}]$$

Taylor Series Approximation

$$\dot{x} = f[\bar{x}, \bar{u}] + \left.\frac{\partial f}{\partial x}\right|_{\bar{u}, \bar{x}} (x - \bar{x}) + \left.\frac{\partial f}{\partial u}\right|_{\bar{u}, \bar{x}} (u - \bar{u})$$

$$y = g[\bar{x}, \bar{u}] + \left.\frac{\partial g}{\partial x}\right|_{\bar{u}, \bar{x}} (x - \bar{x}) + \left.\frac{\partial g}{\partial u}\right|_{\bar{u}, \bar{x}} (u - \bar{u})$$

Approximation in Matrix Form (Lecture 4 Slide 26)

- Introduce variables for small variations:

$$\delta x(t) := x(t) - \bar{x} \qquad \delta u(t) := u(t) - \bar{u} \qquad \delta y(t) := y(t) - \bar{y}$$

- Note that $\delta \dot{x} = \dot{x}$

$$\delta \dot{x} = \left.\frac{\partial f}{\partial x}\right|_{\bar{u},\, \bar{x}} \delta x + \left.\frac{\partial f}{\partial u}\right|_{\bar{u},\, \bar{x}} \delta u \qquad \delta y = \left.\frac{\partial g}{\partial x}\right|_{\bar{u},\, \bar{x}} \delta x + \left.\frac{\partial g}{\partial u}\right|_{\bar{u},\, \bar{x}} \delta u$$

Linearized Version of State Model

$$\delta \dot{x} = A\,\delta x + B\,\delta u$$

$$\delta y = C\,\delta x + D\,\delta u$$

$$\delta x(0) = x_0 - \bar{x}$$

$$A := \left.\frac{\partial f}{\partial x}\right|_{\bar{u},\, \bar{x}} \qquad C := \left.\frac{\partial g}{\partial x}\right|_{\bar{u},\, \bar{x}}$$

$$B := \left.\frac{\partial f}{\partial u}\right|_{\bar{u},\, \bar{x}} \qquad D := \left.\frac{\partial g}{\partial u}\right|_{\bar{u},\, \bar{x}}$$

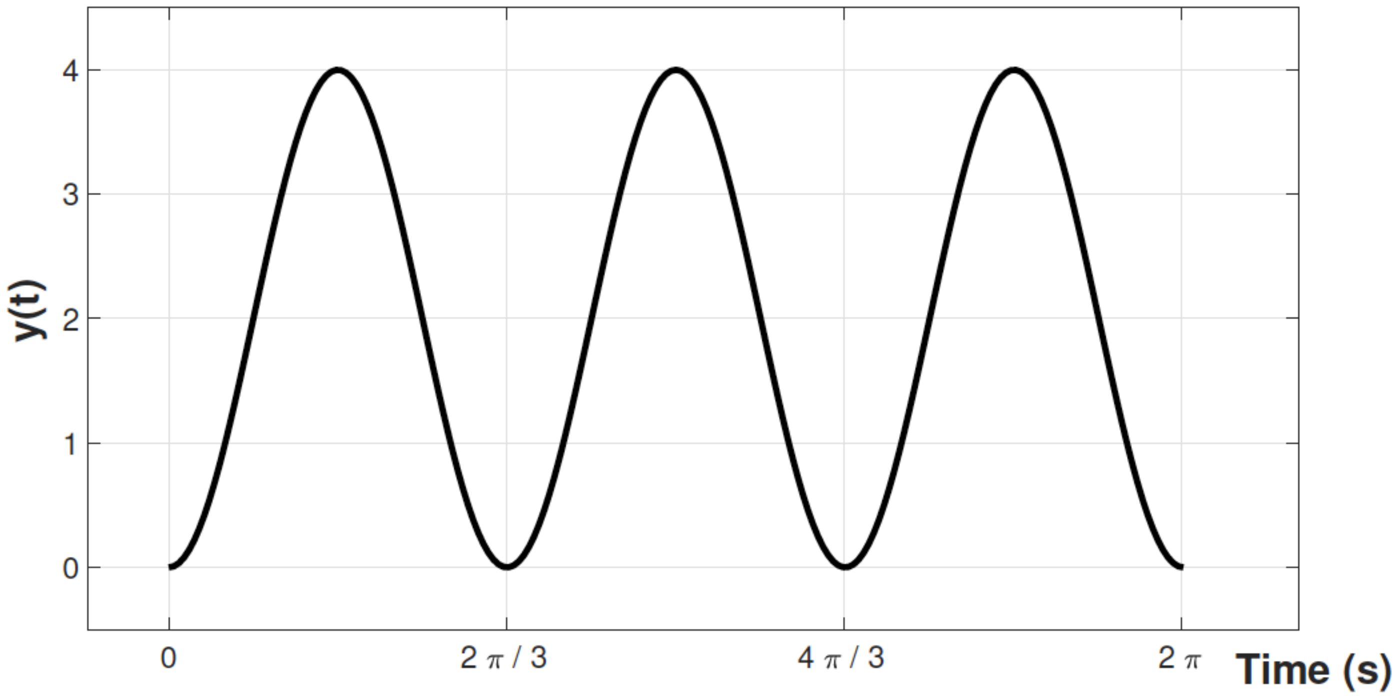
Problem 2

- Transient response characteristics
    - Impulse response
    - Step response
    - Ramp response
    - Arbitrary input function

- Rise time, peak time, settling time
- Location of poles
- Stability

Problem 2

- Finding the transfer function from the unit-step response



Problem 2

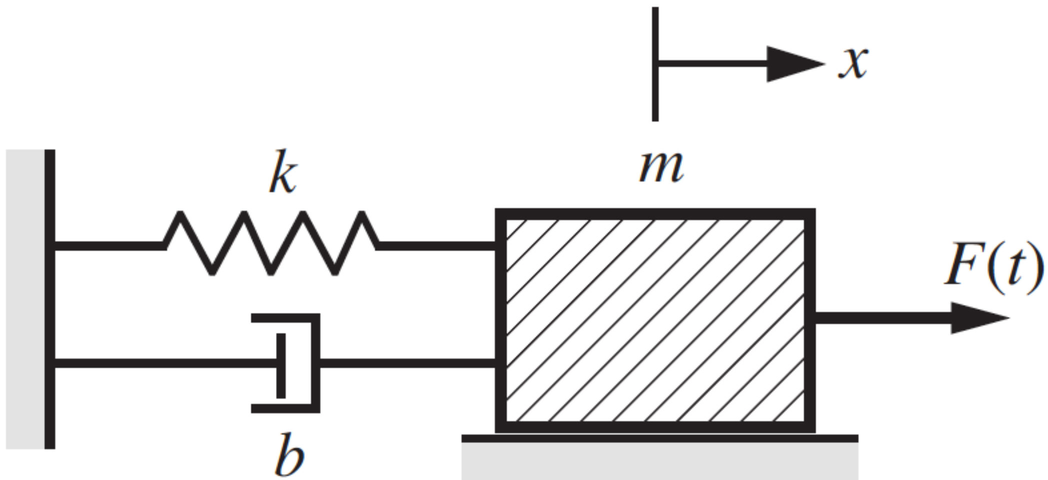
- Pole placement vs the response characteristics

Consider a second order LTI system with two poles p_1 and p_2, one zero z_1, and gain K.

1. (3 points) Write down the transfer function of the system.
2. (4 points) Calculate the unit step response of the system given that $p_1 = -2$, $p_2 = -1$, $z_1 = -5$. and $K = 2$.
3. (6 points) How would the unit step response change if we move the zero from $z_1 = -5$ to $z_1 = -0.5$? Show your work with plots.

Problem 2

- Reconstruction of the system from empirical values



(10 points) Determine the numerical values of m, k, and b so that when the input force is $F(t) = 100\varepsilon(t)$ N

- the mass resides at $x = 2$ cm at the steady-state
- the mass settles to 2% of its final value within 2 seconds
- the maximum percent overshoot of $x(t)$ from its steady-state value is 50%

To ensure you understand the design specifications, first make a rough sketch of the displacement $x(t)$ for $0 \leq t \leq 2$ sec and clearly identify peak time, maximum overshoot, and settling time on the graph. Note that $\varepsilon(t)$ is the unit step function.

Problem 3

- Collection of small questions on Laplace transform and convolution

$$m\ddot{x}(t) = F(t) - kx(t) - f\dot{x}(t) \qquad x(0) = \dot{x}(0) = 0$$

$$m = 2 \text{ kg},\ k = 10 \text{ N/m},\ f = 8 \text{ Ns/m}$$

$$F(t) = \begin{cases} 1 \text{ N} & \text{for } t < 5, \\ 0 & \text{for } t \geq 5. \end{cases}$$

Problem 3

- Collection of small questions on Laplace transform and convolution

Impulse Response

$$g(t) = e^{-t} + e^{-2t}$$

- Find Unit-step Response
  - Convolution
  - Laplace Transform

Problem 3

- Collection of small questions on Laplace transform and convolution

$$\dddot{y}(t) + 8\ddot{y}(t) + 17\dot{y}(t) + 10y(t) = 0$$

$$y(0) = 2,\ \dot{y}(0) = 1,\ \ddot{y}(0) = 0.5$$

- Solve the differential equation and calculate the output

Problem 3

- Collection of small questions on Laplace transform and convolution

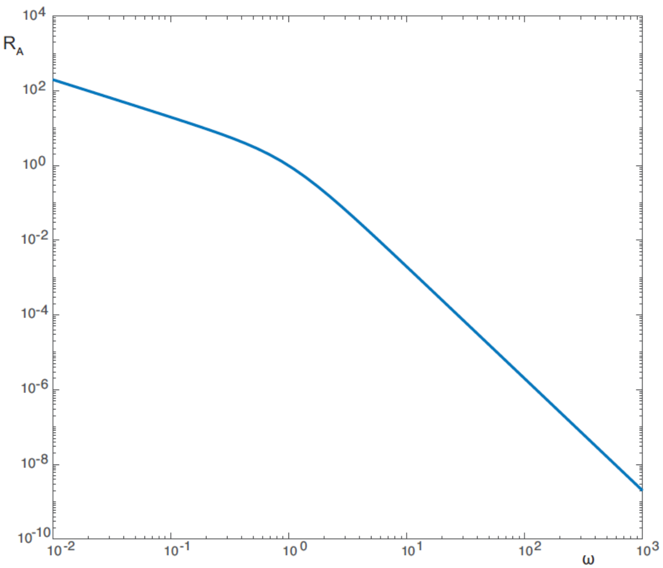
Compute the inverse Laplace transform of

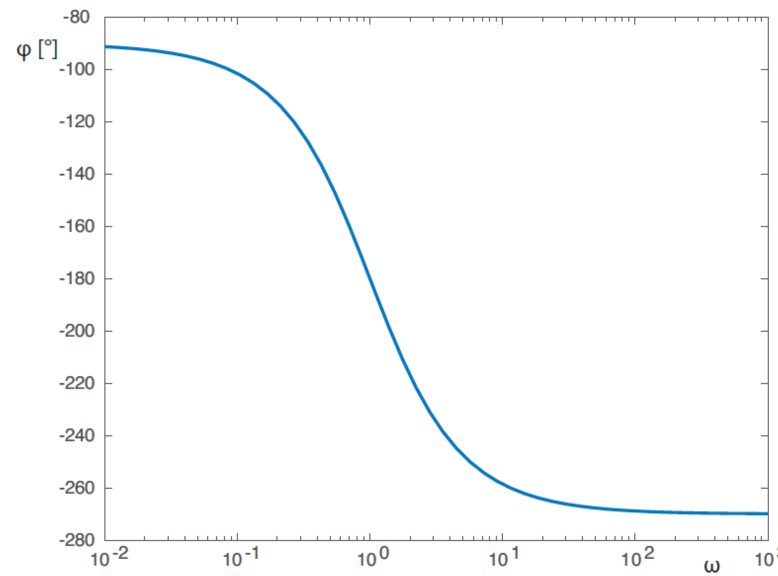
$$G(s) = \frac{3s}{(s^2+1)^2}$$

1. (4 points) Using the following property: $\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$.
2. (6 points) Using the following convolution integral $f_1(t) * f_2(t) = \int_0^t f_1(\tau) f_2(t-\tau) d\tau$.

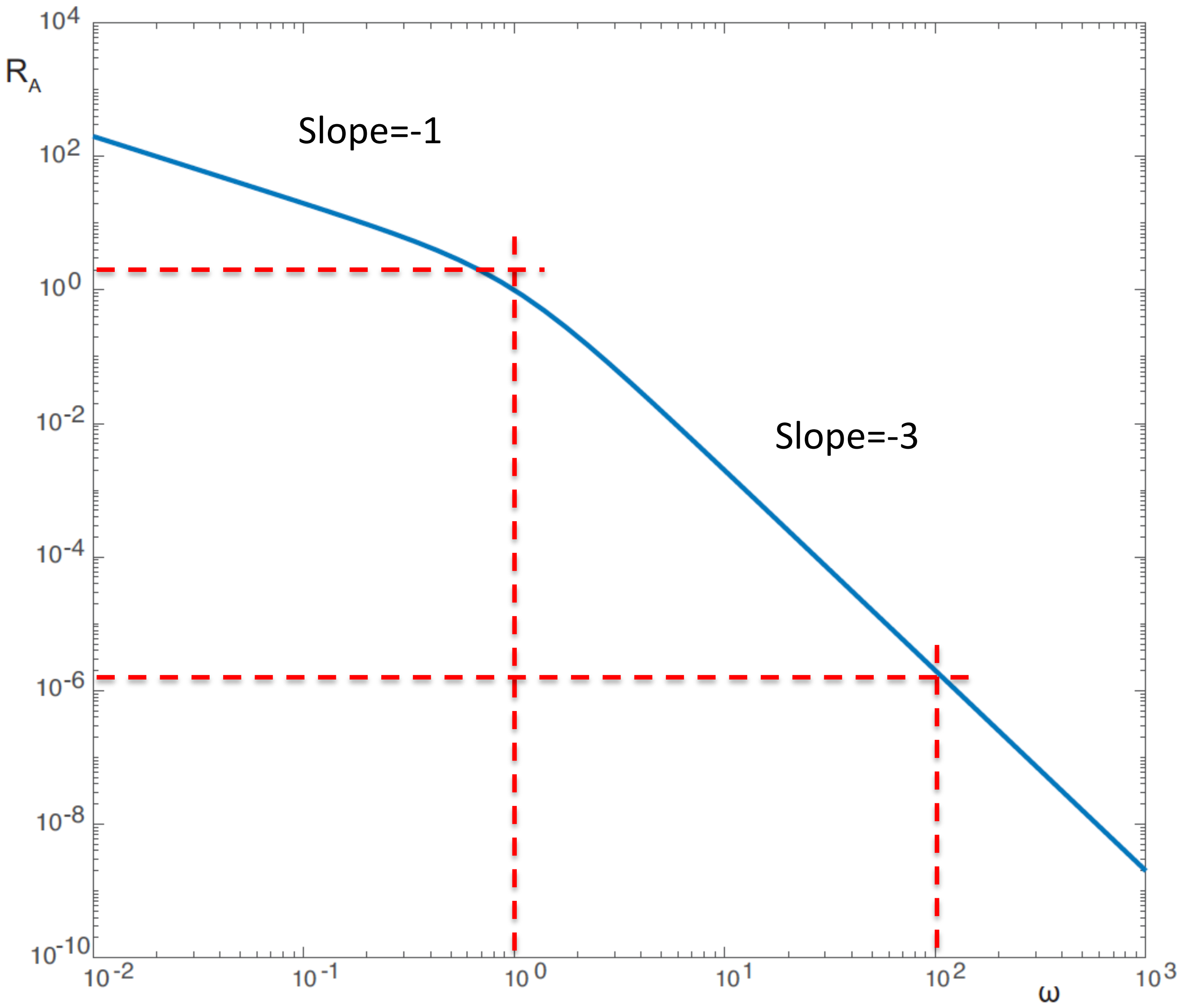
Problem 4 (Frequency Response)

- Bode Plot

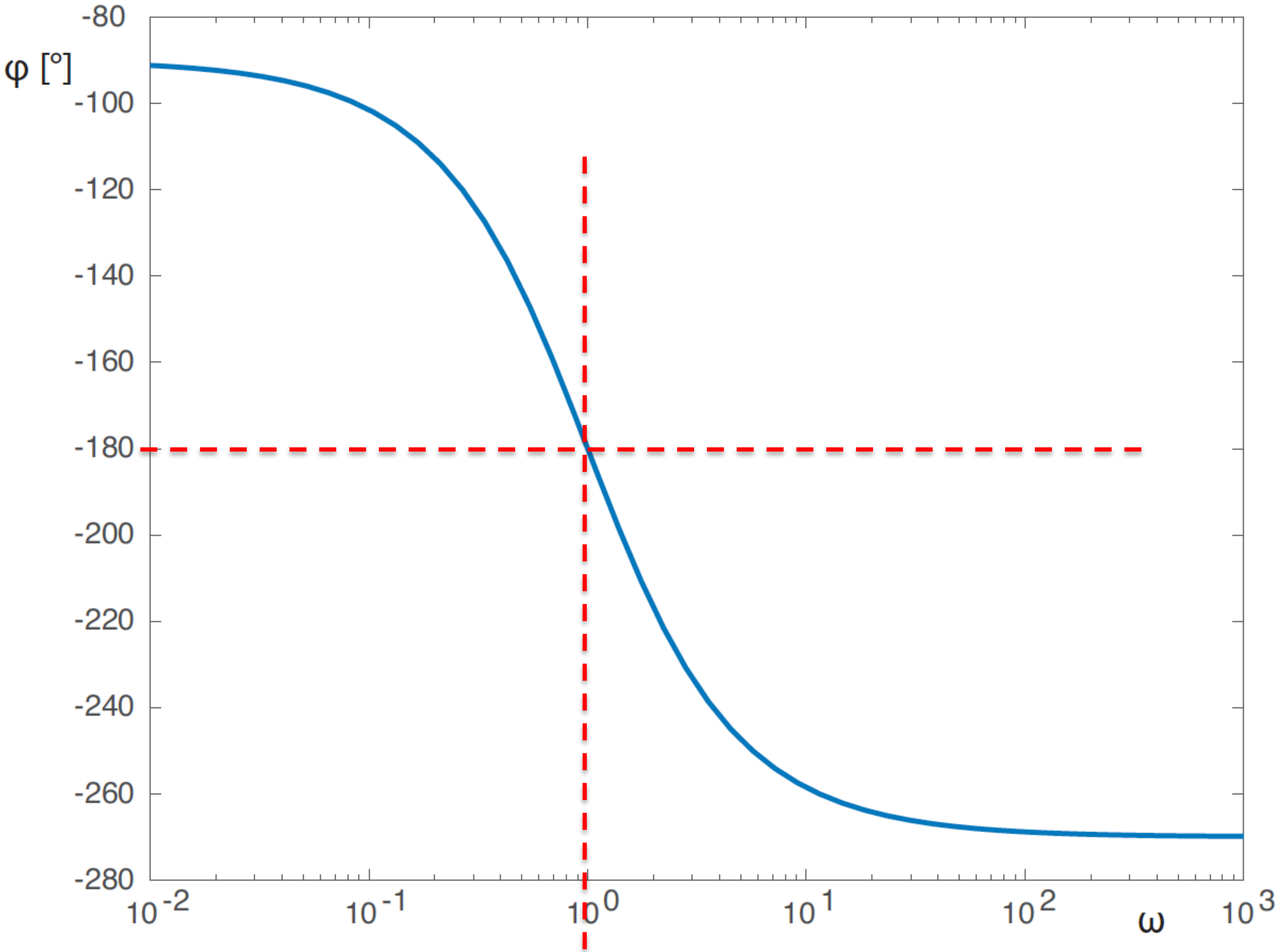




- Calculate transfer function
- Steady-state value
- Filter design



10^4
R_A
Slope=-1
10^2
10^0
10^{-2}
Slope=-3
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-2}
10^{-1}
10^0
10^1
10^2
10^3
ω



-80
φ [°]
-100
-120
-140
-160
-180
-200
-220
-240
-260
-280
10 -2
10 -1
10 0
10 1
10 2
ω
10 3

Problem 4 (Frequency Response)

- Transfer Function

$$G(s) = \frac{K}{s}\frac{1}{(\tau s + 1)^2}$$

$K = 2$ and $\tau = 1$

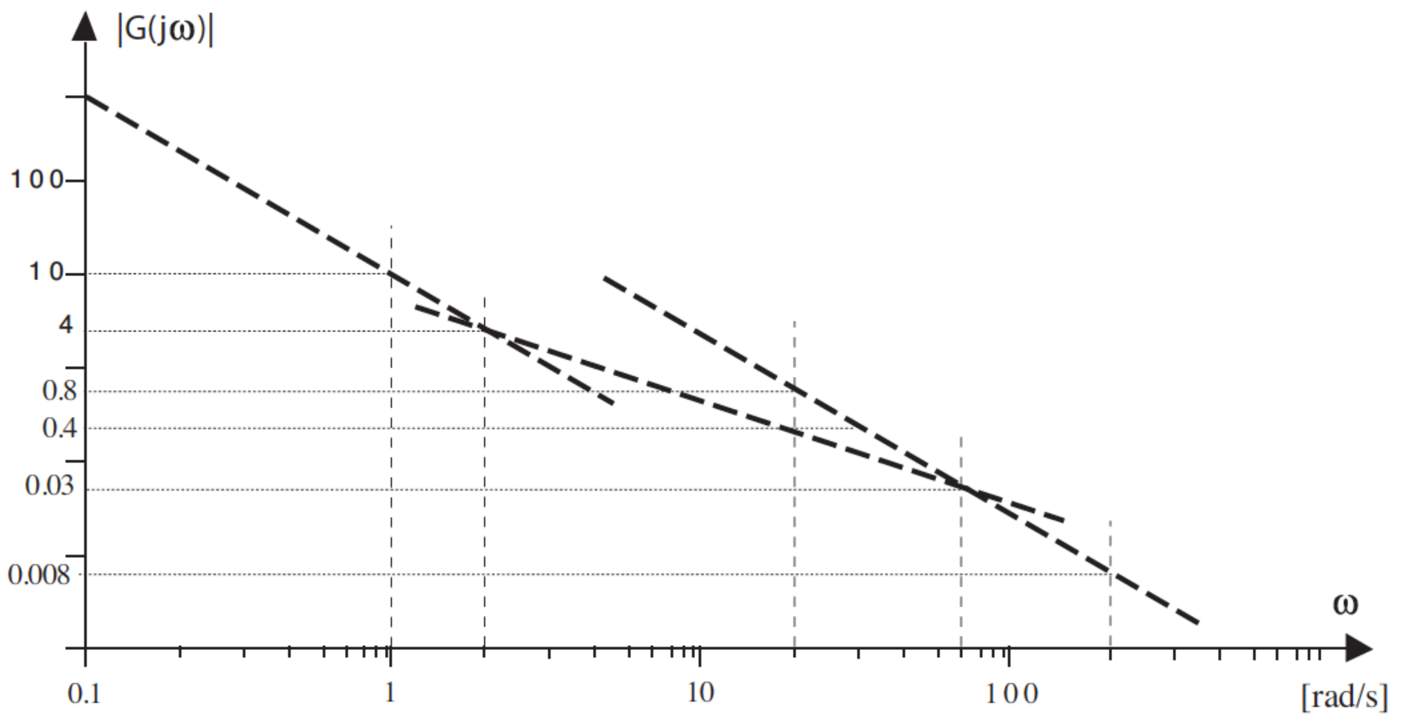
b) Imagine that we want to design a first order filter (denoted by the transfer function $F(s)$) with a zero at -1. The filtered system is expected to have a magnitude of 1 and phase shift of -145 at $\omega = 1$. Find the gain and the time constant of $F(s)$ that would lead to the desired specifications. Note that, the transfer function of the filtered system $G_f(s)$ is simply the product of the transfer function of the original system and the transfer function of the first order filter (i.e. $G_f(s) = G(s)F(s)$).

$$F(s) = \frac{K_f(s+1)}{\tau_f s + 1}$$

$$G_f(s) = \frac{K_f(s+1)}{\tau_f s + 1}\frac{2}{s}\frac{1}{(s+1)^2} = \frac{2K_f}{(\tau_f s + 1)s(s+1)}$$

Problem 4 (Frequency Response)

- Bode Plot (no complex poles or exponential terms)



- Sketch Phase plot
- Transfer function
- Nyquist plot

- Steady-state Output

$$u(t) = 48sin(60t)$$

Another example on Frequency Response

- Transfer Function

$$G(s) = \frac{10^5 s(s+100)}{(s+10)^2(s^2+400s+10^6)}$$

- Sketch Bode plot

Another example on Frequency Response

- Transfer Function

$$G(s) = (s+2)e^{-s}/s(2s+1)$$

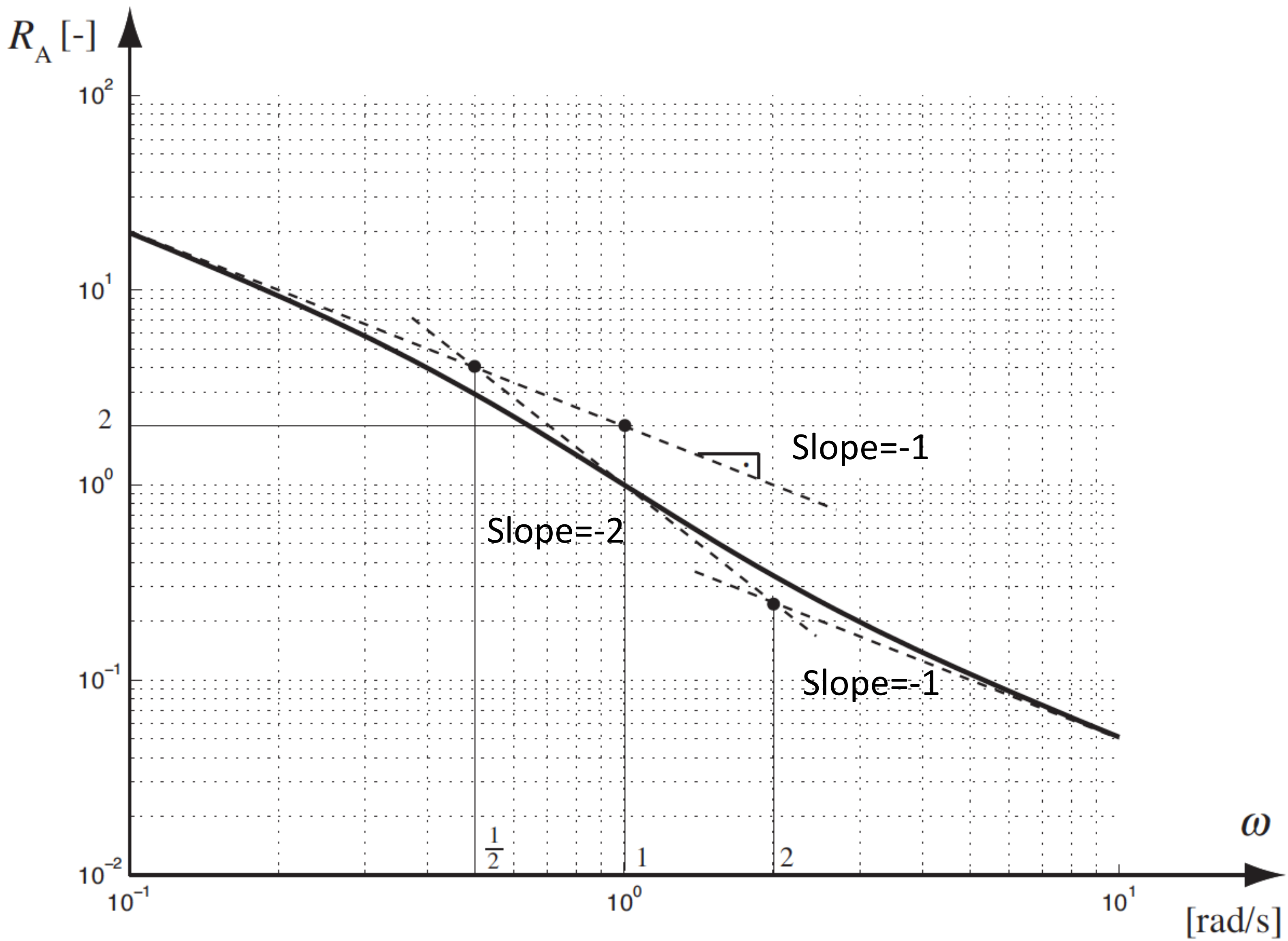
- Sketch Bode plot
- Nyquist Plot
- Calculate steady-state response if the input is

$$u(t) = 2\sin t$$

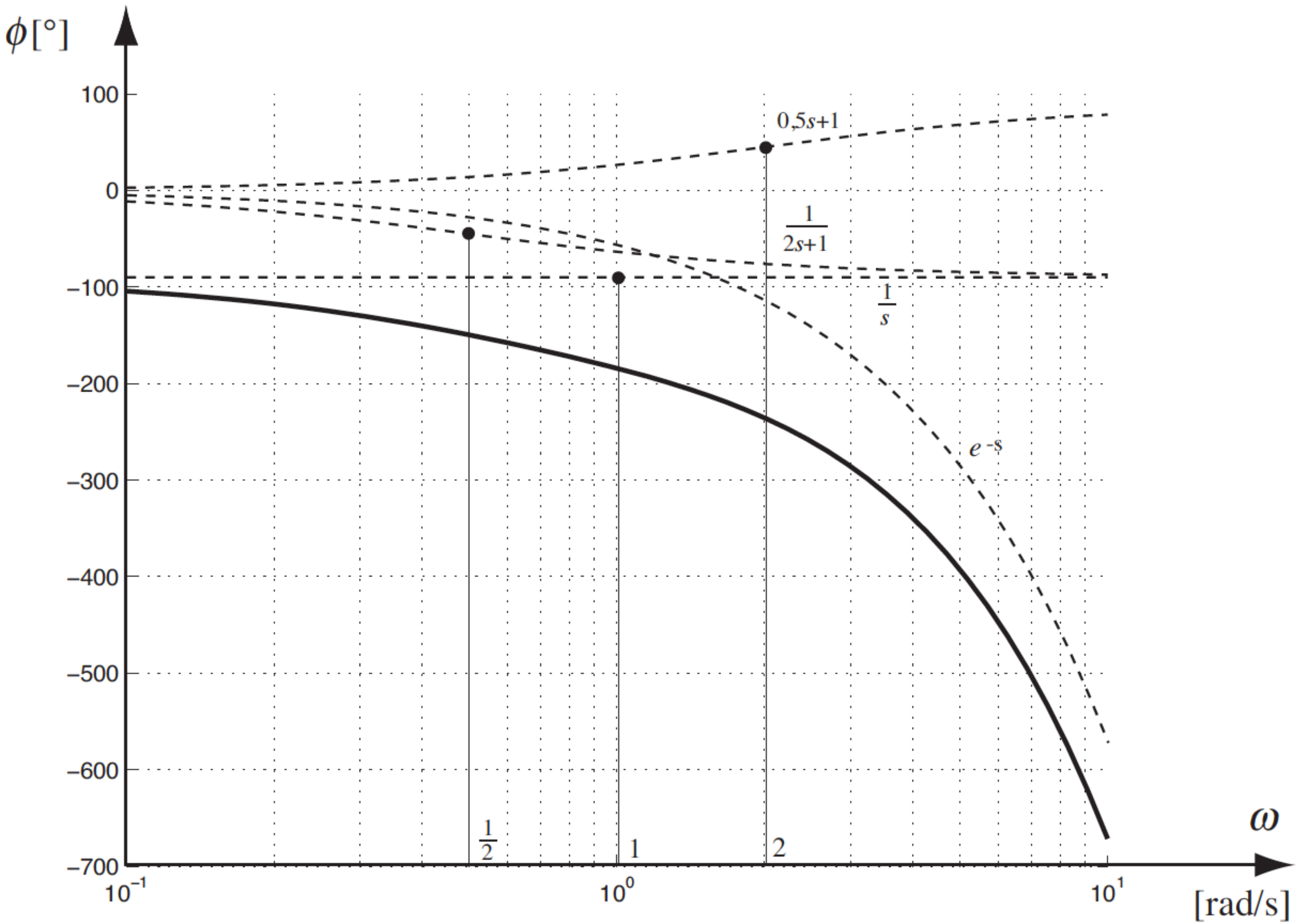
Solution

- Transfer function in **standard form**

$$p_1 = 0\,,\;\; p_2 = -\frac{1}{2}\,,\;\; z_1 = -2 \qquad\qquad G(s) = \frac{2\left(\frac{1}{2}s + 1\right)e^{-s}}{s(2s+1)}$$



R_A [-]
10^2
10^1
2
10^0
10^{-1}
10^{-2}
Slope=-1
Slope=-2
Slope=-1
$\frac{1}{2}$
1
2
ω
10^{-1}
10^0
10^1
[rad/s]



ϕ[°]
100
0
−100
−200
−300
−400
−500
−600
−700
0,5s+1
$\frac{1}{2s+1}$
$\frac{1}{s}$
e^{-s}
$\frac{1}{2}$
1
2
ω
10^{-1}
10^{0}
10^{1}
[rad/s]

Solution

- Input

$$u(t) = 2\sin t$$

$$R_A = \frac{2\sqrt{\frac{1}{4}\omega^2 + 1}}{\omega\sqrt{4\omega^2 + 1}} \qquad R_A(\omega = 1) = \frac{2\sqrt{\frac{1}{4} + 1}}{\sqrt{4 + 1}} = 1$$

$$\varphi = \arctan\left(\frac{1}{2}\omega\right) - \frac{\pi}{2} - \arctan(2\omega) - \omega$$

$$\varphi(\omega = 1) = -3{,}21 \ \text{rad}$$

$$\boxed{\bar{y}(t) = 2\sin(t - 3{,}21)}$$

Solution

- From the Bode Plot

$$R_A(\omega = 1) \cong 1 = \frac{A'}{A} \rightarrow A' = 2$$

$$\varphi(\omega = 1) \cong -200^\circ = -3{,}49 rad$$

$$\bar{y}(t) = A' \sin(t - t') = 2 \sin(t - 3{,}49)$$